# Categoricity of modular and Shimura curves

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## 1 Setup

### 1.1 Shimura varieties

Consider a connected Shimura datum $(G, X)$ consisting of a reductive algebraic group $G$ over $\mathbb{Q}$ and the $G(\mathbb{R})$-conjugacy class $X$ of a homomorphism $h : \mathbb{S} \to G_{\mathbb{R}}$, satisfying the three axioms SV1-SV3 of [4], p.50. Let $X^+$ be a connected component of $X$, let $K$ be a compact open subgroup of $G(\mathbb{A}_f)$ and let $\mathcal{C}$ be a set of representatives for the finite set

$$G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/K,$$

where $G(\mathbb{Q})_+$ is the intersection of $G(\mathbb{Q})$ with the stabiliser of $X^+$ in $G(\mathbb{R})$. For each $g \in \mathcal{C}$ let $\Gamma_g := G(\mathbb{Q})_+ \cap gKg^{-1}$.

The double coset space

$$\mathrm{Sh}_K(G, X) := G(\mathbb{Q})\backslash(X \times (G(\mathbb{A}_f)/K))$$

is homeomorphic to a finite disjoint union

$$\coprod_{g\in\mathcal{C}} \Gamma_g\backslash X^+.$$

When $K$ is sufficiently small, the members of this union may be realised as locally symmetric varieties. That is, $\mathrm{Sh}_K(G, X)$ possesses a unique structure of a quasi-projective variety over $\mathbb{C}$. Moreover, $\mathrm{Sh}_K(G, X)$ is well known to possess a canonical model over its reflex field $E := E(G, X)$.

Let $\Gamma := \Gamma_g$ for some $g \in \mathcal{C}$, which me may as well assume to be the identity in $G(\mathbb{A}_f)$, and consider the connected component $\Gamma\backslash X^+$. Let $S$ be

the corresponding subvariety of the canonical model; it is defined over the maximal abelian extension $E^{\mathrm{ab}}$ of $E$. Let

$$p : X^+ \to S(\mathbb{C})$$

denote the complex-analytic universal cover.

Let $G^{\mathrm{ad}}$ denote the group $G$ modulo its centre. Let

$$G^{\mathrm{ad}}(\mathbb{Q})^+ := G^{\mathrm{ad}}(\mathbb{R})^+ \cap G^{\mathrm{ad}}(\mathbb{Q}).$$

Note that the action of $G(\mathbb{Q})_+$ on $X^+$ factors through $G^{\mathrm{ad}}(\mathbb{Q})^+$. Therefore, the elements of $G^{\mathrm{ad}}(\mathbb{Q})^+$ may be thought of as functions from $X^+$ to itself.

## 1.2 The language

With the situation described in (1.1) in mind, we will consider a first-order language $\mathcal{L}$ for two-sorted structures of the form

$$\mathbf{q} = \langle\langle D, \{g_n\}_{n\in\mathbb{N}}\rangle, \langle V(F), \{V_i\}_{i\in\mathbb{N}}\rangle, q : D \to V(F)\rangle,$$

where

1. $D$ is a set,
2. $\{g_n\}_{n\in\mathbb{N}}$ is a set of countably many unary function symbols,
3. $V(F)$ denotes the $F$-points of a quasi-projective variety $V$, defined over a countable field $k$ of characteristic 0, where $F$ is an algebraically closed field of characteristic 0,
4. $\{V_i\}_{i\in\mathbb{N}}$ is the set of Zariski-closed subsets of $V(F)^n$, defined over $k$, for all $n \in \mathbb{N}$, and
5. $q$ is a function from $D$ to $V(F)$.

A variety $S$, as defined in (1.1), determines an $\mathcal{L}$-structure $\mathbf{p}$; this is the two sorted structure

$$\mathbf{p} := \langle\langle X, G^{\mathrm{ad}}(\mathbb{Q})^+\rangle, \langle S(\mathbb{C}), \{Z_i\}_{i\in\mathbb{N}}\rangle, p : X \to S(\mathbb{C})\rangle,$$

where $\{Z_i\}_{i\in\mathbb{N}}$ is the set of Zariski closed subsets of $S(\mathbb{C})^n$, for all $n \in \mathbb{N}$, that are definable over $E^{\mathrm{ab}}$.

## 1.3 Categoricity

Let $S$ be a variety, as defined in (1.1), and let $\mathbf{p}$ be the corresponding $\mathcal{L}$-structure, as described in (1.2). Let $\mathrm{Th}(\mathbf{p})$ denote the complete first order $\mathcal{L}$-theory of $\mathbf{p}$. Let SF (Standard Fibres) denote the $\mathcal{L}_{\omega_1,\omega}$-axiom

$$\forall x \forall y \left( p(x) = p(y) \to \bigvee_{\gamma \in \Gamma} x = \gamma y \right).$$

We denote by $T_{\omega_1,\omega}(\mathbf{p})$ the union of $\mathrm{Th}(\mathbf{p})$ and SF. Finally, we denote by $T^{\infty}_{\omega_1,\omega}(\mathbf{p})$ the $\mathcal{L}_{\omega_1,\omega}$-sentence comprising $T_{\omega_1,\omega}(\mathbf{p})$ with the additional axiom that the transcendence degree of $F$ is infinite.

Our objective is to determine when the theory $T^{\infty}_{\omega_1,\omega}(\mathbf{p})$ is *categorical* in every uncountable cardinality, by which we mean, given any two $\mathcal{L}$-structures

$$\mathbf{q} := \langle \langle D, G^{\mathrm{ad}}(\mathbb{Q})^+ \rangle, \langle S(F), \{V_i\}_{i \in \mathbb{N}} \rangle, q : D \to S(F) \rangle$$

and

$$\mathbf{q}' := \langle \langle D', G^{\mathrm{ad}}(\mathbb{Q})^+ \rangle, \langle S(F'), \{V_i'\}_{i \in \mathbb{N}} \rangle, q' : D' \to S(F') \rangle$$

of some uncountable cardinality $\kappa$, such that $\mathbf{q}, \mathbf{q}' \models T^{\infty}_{\omega_1,\omega}(\mathbf{p})$, there exists a commutative diagram

$$\begin{array}{ccc} D & \xrightarrow{\varphi} & D' \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q'} \\ S(F) & \xrightarrow{\sigma} & S(F'), \end{array}$$

where $\varphi$ is an isomorphism of $G^{\mathrm{ad}}(\mathbb{Q})^+$-sets and $\sigma$ is a field isomorphism over $E^{\mathrm{ab}}$. However, given such an $\mathcal{L}$-structure $\mathbf{q} \models T^{\infty}_{\omega_1,\omega}(\mathbf{p})$, it follows (see, for example, [8], Fact 3) that the isomorphism types of the top and bottom sorts are uniquely determined by $\kappa$. Hence, in the above diagram, we may replace $D'$ with $D$ and $S(F')$ with $S(F)$, taking $\sigma$ to be an automorphism of $F$ over $E^{\mathrm{ab}}$.

## 1.4 Assumptions

Recall the situation in (1.1). We made reference to the property of a compact open subgroup $K$ of $G(\mathbb{A}_f)$ being 'sufficiently small', having in mind that

the corresponding quotient space $\Gamma\backslash X^+$ should have a realisation as a locally symmetric variety, and such that this structure is unique (see [4], Theorem 3.12 and Corollary 3.16). Recall (see [3], Section 4) the definition of a neat compact open subgroup $K$ of $G(\mathbb{A}_f)$. In particular, such a $K$, and therefore the corresponding $\Gamma$, are torsion free. Hence, in this paper, 'sufficiently small' will be defined as neat. Note that any compact open subgroup $K$ of $G(\mathbb{A}_f)$ contains a neat compact open subgroup $K'$ of $G(\mathbb{A}_f)$ such that the index of $K'$ in $K$ is finite.

# 2 Theory

## 2.1 Special points

Let $[\cdot,\cdot]_K$ denote an equivalence class in the double coset space $\mathrm{Sh}_K(G,X)$, as defined in (1.1). Recall the notion of a special point in $\mathrm{Sh}_K(G,X)$:

**Definition 2.1** *A point $[x,g]_K$ is called special if there exists a torus $T\subset G$, defined over $\mathbb{Q}$, such that $x:\mathbb{S}\to G_{\mathbb{R}}$ factorises through $T_{\mathbb{R}}$. Clearly, the definition does not depend on the choice of $x\in X$ and we also refer to any such $x\in X$ as special.*

The following is a model-theoretic characterisation of special points:

**Theorem 2.2** *Let $x\in X^+$. The following are equivalent:*

1. *The point $x$ is special.*
2. *There exists a $g\in G^{\mathrm{ad}}(\mathbb{Q})^+$ such that $x$ is the unique fixed point of $g$ in $X^+$.*

**Proof.** Firstly, assume that (2) holds for $g\in G^{\mathrm{ad}}(\mathbb{Q})^+$ but replace $g$ with a preimage in $G(\mathbb{Q})$. Let $\mathrm{MT}_x$ be the Mumford-Tate group of $x$ i.e. the smallest algebraic subgroup $M$ of $G$, defined over $\mathbb{Q}$, such that $x$ factors though $M_{\mathbb{R}}$. Since $g$ fixes $x$, $g$ belongs to the centraliser of $\mathrm{MT}_x(\mathbb{Q})$ by [7], Lemma 2.2. Let $X_{\mathrm{MT}_x}$ be the $\mathrm{MT}(\mathbb{R})$-conjugacy class of $x$ and let $X^+_{\mathrm{MT}_x}$ be a connected component contained in $X^+$. Since $g$ fixes every element of $X^+_{\mathrm{MT}_x}$, this component must contain only one element. Thus, the reductive group $\mathrm{MT}_x$ is commutative and, therefore, a torus. Hence, $x$ is special.

Now, assume that (1) holds. Therefore, $x : \mathbb{S} \to G_{\mathbb{R}}$ factors through $T_{\mathbb{R}}$, where $T \subset G$ is maximal torus, defined over $\mathbb{Q}$. Let $\pi : G \to G^{\mathrm{ad}}$ be the natural morphism and let $S := \pi(T)$. Let $K_x$ be the stabiliser of $x$ in $G^{\mathrm{ad}}(\mathbb{R})^+$; it is compact since $G^{\mathrm{ad}}$ is adjoint. Then $S(\mathbb{R})$ is a maximal torus of $K_x$.

Suppose $S(\mathbb{R})$ fixes another point $x' \in X^+$. Then $x' = gx$ for some $g \in G^{\mathrm{ad}}(\mathbb{R})^+$ and $g^{-1}S(\mathbb{R})g$ lies in $K_x$. Since all maximal tori in connected compact lie groups are conjugate, there exists $k \in K_x$ such that

$$(gk)^{-1}S(\mathbb{R})gk = S(\mathbb{R}).$$

Hence, $gk$ lies in the normaliser $N$ of $S(\mathbb{R})$ in $G^{\mathrm{ad}}(\mathbb{R})^+$.

Note that, since $S$ is maximal and $G^{\mathrm{ad}}$ is reductive, the centraliser of $S$ in $G^{\mathrm{ad}}$ is $S$ itself. Thus, the quotient $N/S(\mathbb{R})$ is the Weyl group of $S(\mathbb{R})$ in $G^{\mathrm{ad}}(\mathbb{R})^+$ and, therefore, finite. Hence, $N$ is compact and, since all maximal compact subgroups in connected real lie groups are conjugate, it must be contained in $hK_xh^{-1}$ for some $h \in G^{\mathrm{ad}}(\mathbb{R})^+$.

Therefore, $h^{-1}S(\mathbb{R})h \subset K_x$ and, since all maximal tori in connected compact lie groups are conjugate, there exists a $k \in K_x$ such that

$$(hk)^{-1}S(\mathbb{R})(hk) = S(\mathbb{R}).$$

Thus, $hk \in N$ and so $h = nk$ for some $n \in N$. Therefore, $hK_xh^{-1} = nK_xn^{-1}$ and so $N \subset K_x$. Thus, $g \in K_x$ and $x' = x$, which proves that $x$ is the unique point of $X^+$ fixed by $S(\mathbb{R})$.

Let $s \in S(\mathbb{Q})$ be a regular element. By this we mean an element $s \in S(\mathbb{Q})$ such that the centraliser $C_{G^{\mathrm{ad}}}(s)$ of $t$ in $G^{\mathrm{ad}}$ is equal to the centraliser $C_{G^{\mathrm{ad}}}(S)$ of $S$ in $G^{\mathrm{ad}}$. These are precisely the elements of $S(\overline{\mathbb{Q}})$ that do not lie in $\ker(\alpha)$ for any of the finitely many roots $\alpha$ of $S$. Note that the (finite) union of the Galois conjugates of these subvarieties is a proper subvariety of $S$, defined over $\mathbb{Q}$. Hence, the complement is open in $S$, whereas $S(\mathbb{Q})$ is dense in $S$. Thus, we can indeed find such a $s \in S(\mathbb{Q})$.

Suppose that $s$ fixes a point $x' : \mathbb{S} \to G_{\mathbb{R}}$ in $X^+$, so that $\mathrm{ad}(s)(t) = t$ for all $t \in x'(\mathbb{S})(\mathbb{R})$. Therefore, $\pi(t)s\pi(t)^{-1} = s$ for all $t \in x'(\mathbb{S})(\mathbb{R})$ and so $\pi(x'(\mathbb{S}))(\mathbb{R})$ is contained in $C_{G^{\mathrm{ad}}}(s)(\mathbb{R})$, which is equal to $C_{G^{\mathrm{ad}}}(S)(\mathbb{R})$, since $s$ is regular. Therefore, $S(\mathbb{R})$ fixes $x'$, which implies that $x' = x$.

$\square$

Consider the $\mathcal{L}$-structure $\mathbf{p}$ corresponding to $S$, as described in (1.2). The upshot of the above characterisation is that the set of special points in $X^+$

belongs to the definable closure of the empty set in the sort $\langle X^+, G^{\mathrm{ad}}(\mathbb{Q})^+\rangle$. Thus, any $\mathcal{L}$-structure $\mathbf{q}$ has a well-defined notion of special point.

## 2.2 Hodge-generic points

Following on from the previous section, recall the notion of a Hodge-generic point in $\mathrm{Sh}_K(G, X)$:

**Definition 2.3** *A point $[x, g]_K$ is called Hodge-generic if $x : \mathbb{S} \to G_{\mathbb{R}}$ does not factor through $H_{\mathbb{R}}$ for any proper algebraic subgroup $H \subset G$ defined over $\mathbb{Q}$. Again, the definition is clearly independent of the choice of $x \in X$ and we also refer to any such $x \in X$ as Hodge-generic.*

The following property of Hodge-generic points follows immediately from [7], Lemma 2.2:

**Proposition 2.4** *Let $x \in X^+$ be Hodge-generic and let $g \in G^{\mathrm{ad}}(\mathbb{Q})^+$ fix $x$. Then $g$ is the identity.*

Therefore, as in the previous section, any $\mathcal{L}$-structure $\mathbf{q}$ has a well-defined notion of Hodge-generic point.

**Definition 2.5** *Let $S$ be a variety as defined in (1.1). If every point in $S$ is either special or Hodge-generic, we define $S$ to be bi-typal.*

## 2.3 Realising finite subsets of a type

Consider the setup described in (1.1). The following fact constitutes the key statement regarding the theory of the Hodge-generic points in $X^+$:

**Lemma 2.6** *Given $g_1, ..., g_n \in G^{\mathrm{ad}}(\mathbb{Q})^+$, for any $n \in \mathbb{N}$, the image of the map $f : X^+ \to S^n(\mathbb{C})$, defined by*

$$f(x) := (p(g_1 x), ..., p(g_n x)),$$

*is an algebraic subvariety of $S^n(\mathbb{C})$ defined over $E^{\mathrm{ab}}$.*

**Proof.** Let $K_g := g_1 K g_1^{-1} \cap \cdots \cap g_n K g_n^{-1}$. The image of $f$ is an irreducible component of the image of the map

$$\mathrm{Sh}_{K_g}(G, X) \to \mathrm{Sh}_{K^n}(G^n, X^n)$$

induced by the morphism $G \to G^n$, defined by

$$g \mapsto (g_1 g g_1^{-1}, ..., g_n g g_n^{-1}).$$

By [4], Theorem 13.6, this map is defined over $E$. Since the connected components of $\mathrm{Sh}_{K_g}(G, X)$ are defined over $E^{\mathrm{ab}}$, the result follows.

□

Now, let $\mathbf{p}$ be the $\mathcal{L}$-structure, as described in (1.2) and suppose that we have two models of $\mathrm{Th}(\mathbf{p})$, which we denote

$$\mathbf{q} := \langle \langle D, G^{\mathrm{ad}}(\mathbb{Q})^+ \rangle, \langle S(F), \{V_i\}_{i \in \mathbb{N}} \rangle, q : D \to S(F) \rangle$$

and

$$\mathbf{q}' := \langle \langle D, G^{\mathrm{ad}}(\mathbb{Q})^+ \rangle, \langle S(F), \{V_i\}_{i \in \mathbb{N}} \rangle, q' : D \to S(F) \rangle.$$

The following corollary will demonstrate that, in any model of $\mathrm{Th}(\mathbf{p})$, one can realise any finite subset of certain types.

**Corollary 2.7** *Let $x \in D$, let $g_1, ..., g_n$ be finitely many elements of $G^{\mathrm{ad}}(\mathbb{Q})^+$ and let $\sigma \in \mathrm{Aut}(K/E^{\mathrm{ab}})$ for some field extension $K$ of $E^{\mathrm{ab}}$. Then there exists an $x' \in D$ such that the tuple*

$$(q'(g_1 x'), ..., q'(g_n x')) \in S^n(F)$$

*is a realisation of*

$$\mathrm{qftp}_{\mathbf{S}}((q(g_1 x), ..., q(g_n x))/K)^{\sigma}.$$

**Proof.** Consider the subvariety of $S^n(\mathbb{C})$ defined in Lemma 2.6. Since it is defined over $E^{\mathrm{ab}}$, the $F$-valued points correspond to one of the $V_i$ in $S^n(F)$. Since $\mathbf{q}$ is a model of $\mathrm{Th}(\mathbf{p})$, the sentence

$$\forall x \in D \; (q(g_1 x), ..., q(g_n x)) \in V_i$$

is true in $\mathbf{q}$. The quantifier-free type

$$\mathrm{qftp}_{\mathbf{S}}((q(g_1x), ..., q(g_nx))/K)$$

is determined by the smallest algebraic subvariety of $S^n(F)$, defined over $K$, containing the tuple in question. Therefore, it is contained in $V_i$ and, since $V_i^\sigma = V_i$, so is the algebraic subvariety that determines

$$\mathrm{qftp}_{\mathbf{S}}((q(g_1x), ..., q(g_nx))/K)^\sigma.$$

On the other hand, since $\mathbf{q}'$ is also a model of $\mathrm{Th}(\mathbf{p})$, the sentence

$$\forall z \in V_i \; \exists x' \in D \; (q'(g_1x'), ..., q'(g_nx')) = z$$

is true in $\mathbf{q}'$. □

## 2.4 Description of the types

Let $S$ be a variety, as defined in (1.1), and let $\mathbf{p}$ be the corresponding $\mathcal{L}$-structure, as described in (1.2). Suppose that

$$\mathbf{q} := \langle \mathbf{D}, \mathbf{S} \rangle := \langle \langle D, G^{\mathrm{ad}}(\mathbb{Q})^+ \rangle, \langle S(F), \{V_i\}_{i \in \mathbb{N}} \rangle, q : D \to S(F) \rangle$$

is a model of $\mathrm{Th}(\mathbf{p})$.

**Proposition 2.8** *Let $\mathcal{L}_{\mathbf{D}}$ and $\mathcal{L}_{\mathbf{S}}$ denote the natural languages for $\mathbf{D}$ and $\mathbf{S}$, respectively. Let $x_1, ..., x_n \in D$ be a collection of Hodge-generic points and suppose that $\varphi$ is a quantifier-free $\mathcal{L}_{\mathbf{D}}$-formula. Then there exists a quantifier-free $\mathcal{L}_{\mathbf{S}}$-formula $\varphi'$ such that*

$$\mathbf{D} \models \varphi(x_1, ..., x_n) \iff \mathbf{S} \models \varphi'(p(x_1), ..., p(x_n)).$$

**Proof.** Recall Proposition 2.4; since the $x_i$ are Hodge generic, the only quantifier-free $\mathcal{L}_{\mathbf{D}}$-formulae that they can satisfy are those of the form $x_i = gx_j$, for $i \neq j$ and $g \in G^{\mathrm{ad}}(\mathbb{Q})^+$. These formulae are equivalent to stating that the $(z_i, z_j) \in S^2(F)$ lie on subvarieties of the form described in Lemma 2.6. Since these subvarieties are defined over $E^{\mathrm{ab}}$, the claim follows. □

Consider the quantifier-free type of a $\mathbf{D}$-tuple $(x_1, ..., x_n)$ of Hodge-generic points in $\mathbf{q}$. By Proposition 2.8, it is equivalent to the quantifier-free $\mathcal{L}_{\mathbf{S}}$-formulae satisfied by finite $\mathbf{S}$-tuples of elements $q(gx_i)$ for all $g \in G^{\mathrm{ad}}(\mathbb{Q})^+$ and $i = 1, ..., n$.

On the other hand, by Theorem 2.1, the quantifier-free type of a special point $x \in D$ is determined by the formula $gx = x$ for some $g \in G^{\mathrm{ad}}(\mathbb{Q})^+$ such that $x$ is the unique element in $D$ to satisfy this formula.

## 2.5 Axiomatisation

Let $S$ be as in (1.1) and let $\mathbf{p}$ be the corresponding $\mathcal{L}$-structure, as described in (1.2). Consider the subvarieties defined over $E^{\mathrm{ab}}$ and described in Lemma 2.6. For each such subvariety $Z$ and the corresponding ordered tuple $(g_1, ..., g_n)$ of elements in $G^{\mathrm{ad}}(\mathbb{Q})^+$, let $\Psi_{(g_1,...,g_n)}$ be the union of the two first order sentences

$$\forall x \in X^+ \ (p(g_1 x), ..., p(g_n x)) \in Z$$

and

$$\forall z \in Z \ \exists x \in X^+ \ (p(g_1 x), ..., p(g_n x)) = z$$

occurring in $\mathrm{Th}(\mathbf{p})$.

Let $T(\mathbf{p})$ denote the union of the following sentences contained in $\mathrm{Th}(\mathbf{p})$:

1. the complete first order theory of the structure $\langle X^+, G^{\mathrm{ad}}(\mathbb{Q})^+ \rangle$ in its natural language,
2. the complete first order theory of the structure $\langle S(\mathbb{C}), \{Z_i\}_{i \in \mathbb{N}} \rangle$ in its natural language,
3. $\Psi_{(g_1,...,g_n)}$ for every finite length ordered tuple, as above, and
4. the complete type of every special point $x \in X^+$.

## 2.6 Quantifier elimination and completeness

Let $S$ be a bi-typal variety, as defined in (1.1), and let $\mathbf{p}$ be the corresponding $\mathcal{L}$-structure, as described in (1.2).

**Proposition 2.9** *Suppose that*

$$\mathbf{q} := \langle \mathbf{D}, \mathbf{S} \rangle := \langle \langle D, G^{\mathrm{ad}}(\mathbb{Q})^+ \rangle, \langle S(F), \{V_i\}_{i \in \mathbb{N}} \rangle, q : D \to S(F) \rangle$$

*and*

$$\mathbf{q}' := \langle \mathbf{D}', \mathbf{S}' \rangle := \langle \langle D, G^{\mathrm{ad}}(\mathbb{Q})^+ \rangle, \langle S(F), \{V_i\}_{i \in \mathbb{N}} \rangle, q' : D \to S(F) \rangle$$

*are $\omega$-saturated models of $T(\mathbf{p})$ and*

$$\rho : D \cup S(F) \to D \cup S(F)$$

*is a partial isomorphism with finitely generated domain $U$. Then, given any $\alpha \in D \cup S(F)$, $\rho$ extends to the substructure generated by $U \cup \{\alpha\}$.*

**Proof.** By definition, $U$ is the union of $U_{\mathbf{D}} := U \cap D$ and $U_{\mathbf{S}} := U \cap S(F)$, where $U_{\mathbf{D}}$ is the union of the $G^{\mathrm{ad}}(\mathbb{Q})^+$-orbits of finitely many $x \in D$ and $U_{\mathbf{S}}$ is $S(K)$ for some field $K$ generated by the co-ordinates of the images of these orbits in $S(F)$ along with finitely many other points in $S(F)$. Note that $\rho$ consists of an injection $\varphi : U_{\mathbf{D}} \to D$ of $G^{\mathrm{ad}}(\mathbb{Q})^+$-sets and an element $\sigma \in \mathrm{Aut}(K/E^{\mathrm{ab}})$ acting on $S(K)$.

First consider the case $z \in S(F)$. We may assume that $z \notin \varphi(U_{\mathbf{D}})$ since, otherwise, we have $z \in U_{\mathbf{S}}$. Then $\mathrm{qftp}_{\mathbf{q}}(z/U)$ is determined by $\mathrm{qftp}_{\mathbf{S}}(z/K)$. In this case, we can extend $\rho$ by choosing a realisation of $\mathrm{qftp}_{\mathbf{S}}(z/K)^{\sigma}$.

Now consider the case $x \in D$ such that $x \notin U_{\mathbf{D}}$. Let $C$ be a finite subset of elements in $S(F)$ whose coordinates generate $K$ over the coordinates of $q(U_{\mathbf{D}})$ adjoined to $E^{\mathrm{ab}}$. Thus, we can replace $E^{\mathrm{ab}}$ with the extension generated by the coordinates of these elements and, henceforth, assume that $D_{\mathbf{S}}$ is generated by $q(U_{\mathbf{D}})$.

If $x$ is special then there is only one choice for $\varphi(x)$. Therefore, assume that $x$ is Hodge-generic. Note that $\rho(\mathrm{qftp}_{\mathbf{q}}(x/U))$ is determined by the union of all $\mathrm{qftp}_{\mathbf{S}}((q(g_1x), ..., q(g_nx))/K)^{\sigma}$ over all tuples $(g_1, ..., g_n)$ of elements in $G^{\mathrm{ad}}(\mathbb{Q})^+$, for all $n \in \mathbb{N}$. By Corollary 2.7, every finite subset of this type is realisable in any model of $T(\mathbf{p})$. Therefore, by compactness, the type is consistent and, since $\mathbf{q}'$ is $\omega$-saturated, it has a realisation $x' \in D$. $\square$

**Corollary 2.10** *The theory $T(\mathbf{p})$ is complete and model complete. It also has quanitifier elimination and is superstable.*

**Remark 2.11** *By quantifier elimination it is now clear that, in any model of* $\mathrm{Th}(\mathbf{p})$*, the definable closure of the empty set is the set of special points.*

# 3 Galois action

Let $S$ be a variety, as defined in (1.1). For any $n \in \mathbb{N}$, consider the the image of the map $X^+ \to S^n(\mathbb{C})$ given by

$$x \mapsto (p(g_1x), ..., p(g_nx))$$

for a tuple $g := (g_1, ..., g_n)$ of distinct elements in $G^{\mathrm{ad}}(\mathbb{Q})^+$. By Lemma 2.6, this is an algebraic subvariety $Z_g \subset S^n(\mathbb{C})$, defined over $E^{\mathrm{ab}}$, and biholomorphic to $\Gamma_g \backslash X^+$, where

$$\Gamma_g := g_1^{-1}\Gamma g_1 \cap \cdots \cap g_n^{-1}\Gamma g_n.$$

Now, consider another tuple $g' := (g'_1, ..., g'_m)$ such that there exists $\alpha \in G^{\mathrm{ad}}(\mathbb{Q})^+$ satisfying $\Gamma_{\alpha g} := \alpha \Gamma_g \alpha^{-1} \subset \Gamma_{g'}$. Then, the holomorphic map $\alpha : X^+ \to X^+$ induces a finite étale covering $Z_g \to Z_{g'}$ via

$$\Gamma_g \backslash X^+ \to \Gamma_{g'} \backslash X^+,$$

whose fibres correspond to the $\Gamma_{g'}$-set $\Gamma_{\alpha g} \backslash \Gamma_{g'} \cong \Gamma_g \backslash \Gamma_{\alpha^{-1} g'}$. Thus, if $\Gamma_{\alpha g}$ is a normal subgroup of $\Gamma_{g'}$, then the fibres are $\Gamma_{\alpha g} \backslash \Gamma'_g$-torsors and multiplication by elements of $\Gamma_{g'}$ are covering automorphisms. On $Z_g$, these are given by regular maps, defined over $E^{\mathrm{ab}}$. Thus, if $z \in Z_{g'}$ has coordinates in an extension $K$ of $E^{\mathrm{ab}}$, the action of $\mathrm{Aut}(\mathbb{C}/K)$ on the fibre above $z$ commutes with the action of $\Gamma_{\alpha g} \backslash \Gamma_{g'}$ and we obtain (see, for example, [6], Section 6) a continuous homomorphism

$$\mathrm{Aut}(\mathbb{C}/K) \to \Gamma_{\alpha g} \backslash \Gamma_{g'}.$$

In particular, we obtain an inverse system, comprising finite étale covers of $S(\mathbb{C})$, indexed by tuples $g = (g_1, ..., g_n)$ of elements in $G^{\mathrm{ad}}(\mathbb{Q})^+$, for all $n \in \mathbb{N}$, such that there exists $\alpha \in G^{\mathrm{ad}}(\mathbb{Q})^+$, such that $\Gamma_{\alpha g}$ is a normal subgroup of $\Gamma$. Note that, for any $g \in G(\mathbb{Q})$, the double coset $\Gamma g \Gamma$ is the disjoint union of finitely many single cosets $\Gamma g_i$ for $g_i \in G(\mathbb{Q})$ (see [5], Lemma 5.29). Hence, this system of normal subgroups is coinitial in the system coming from all tuples. In the limit, the fibre above a point in $S(\mathbb{C})$ is a $\overline{\Gamma}$-torsor, where $\overline{\Gamma} := \varprojlim \Gamma_{\alpha g} \backslash \Gamma$, and this action is given by compatible covering automorphisms on each of the $Z_g$, defined over $E^{\mathrm{ab}}$. Then, if $z \in S(\mathbb{C})$ has coordinates in an extension $K$ of $E^{\mathrm{ab}}$, the action of $\mathrm{Aut}(\mathbb{C}/K)$ on this fibre is given by a continuous homomorphism

$$\mathrm{Aut}(\mathbb{C}/K) \to \overline{\Gamma}.$$

Similarly, if we take a tuple of points $z := (z_1, ..., z_m) \in S^m(\mathbb{C})$, with coordinates in an extension $K$ of $E^{\mathrm{ab}}$, then the action of $\mathrm{Aut}(\mathbb{C}/K)$ on the fibre above z in the $m$-fold product of the above limit is given by a continuous homomorphism

$$\mathrm{Aut}(\mathbb{C}/K) \to \overline{\Gamma}^m.$$

# 4 Necessary conditions

Let $S$ be a bi-typal subvariety, as defined in 1.1. Consider the inverse system of varieties

$$\left(\Gamma_g \backslash X^+\right)_g$$

from the previous section, where $g$ varies over $n$-tuples of elements in $G^{\mathrm{ad}}(\mathbb{Q})^+$ such that there exists an $\alpha \in G^{\mathrm{ad}}(\mathbb{Q})^+$ such that $\Gamma_{\alpha g}$ is a normal subgroup of $\Gamma$.

Note that this system carries an action of $G^{\mathrm{ad}}(\mathbb{Q})^+$: let $\alpha \in G^{\mathrm{ad}}(\mathbb{Q})^+$ and let $g$ be a tuple, as above. Then the action of $\alpha$ on $X^+$ defines a regular map

$$\Gamma_g \backslash X^+ \to \alpha \Gamma_g \alpha^{-1} \backslash X^+.$$

We denote the inverse limit of the above system by $\mathrm{Sh}^\circ$ and denote an equivalence class in $\Gamma_g \backslash X^+$ by $[\cdot]_g$. A point in $\mathrm{Sh}^\circ$ consists of a compatible collection of points $[x_g]_g$ in each of the $\Gamma_g \backslash X^+$, so that the above action of $G^{\mathrm{ad}}(\mathbb{Q})^+$ is

$$[x_g]_g \mapsto [\alpha x_g]_{\alpha g}.$$

Note that, if $[x_g]_g$ is Hodge-generic, then all points in a compatible collection containing $[x_g]_g$ must be Hodge-generic. Thus, we have a well-defined notion of Hodge-generic points on $\mathrm{Sh}^\circ$.

Now consider a Hodge-generic point in $\mathrm{Sh}^\circ$ i.e. a compatible collection $[x_g]_g$. We would like to show that there exists a model

$$\mathbf{q} := \langle \langle D, G^{\mathrm{ad}}(\mathbb{Q})^+ \rangle, \langle S(\mathbb{C}), \{Z_i\}_{i\in\mathbb{N}} \rangle, q : D \to S(\mathbb{C}) \rangle$$

of $T_{\omega_1,\omega}(\mathbf{p})$, of cardinality $2^{\aleph_0}$, such that there exists an $x \in D$, such that

$$(q(g_1 x), ..., q(g_n x)) \in Z_g$$

corresponds to $[x_g]_g \in \Gamma_g \backslash X^+$ for all tuples $g = (g_1, ..., g_n)$ under the embeddings

$$[x]_g \mapsto [g_1 x, ..., g_n x] : \Gamma_g \backslash X^+ \hookrightarrow \left(\Gamma \backslash X^+\right)^n.$$

Note that we have an embedding

$$x \mapsto ([x]_g)_g : X^+ \hookrightarrow Sh^\circ,$$

since the intersection $\cap_{\alpha g}\Gamma_{\alpha g}$ is clearly trivial. Therefore, we denote by $D$ the union of $G^{\mathrm{ad}}(\mathbb{Q})^{+}\cdot x$ and the points in $X^{+}$ lying above points in $S(\mathbb{C})$ not contained in the image of this orbit under the natural map

$$\pi : \mathrm{Sh}^{\circ} \to S(\mathbb{C}).$$

We complete the description of $\mathbf{q}$ by letting $q$ be the restriction of $\pi$ to $D$.

Let $g = (g_1, ..., g_n)$ be a tuple. Then, the $i^{\mathrm{th}}$-coordinate of

$$[x_g]_g \in \left(\Gamma\backslash X^{+}\right)^n,$$

under the above embedding, is $[g_i x_g]$, corresponding to $q(g_i x)$. Note that, by the completeness of $T(\mathbf{p})$ (Corollary 2.10), it is easy to see that $\mathbf{q} \models \mathrm{Th}(\mathbf{p})$, and $\mathbf{q} \models \mathrm{SF}$ is obvious.

The above construction easily generalises to the following:

**Proposition 4.1** *Let $m \in \mathbb{N}$ and consider a set of Hodge-generic points*

$$\{x_i = ([x_{i,g}]_g)_g\}_{i=1}^m \subset \mathrm{Sh}^{\circ}$$

*with disjoint $G^{\mathrm{ad}}(\mathbb{Q})^{+}$-orbits. Then, there exists a model*

$$\mathbf{q} := \langle\langle D, G^{\mathrm{ad}}(\mathbb{Q})^{+}\rangle, \langle S(\mathbb{C}), \{Z_i\}_{i\in\mathbb{N}}\rangle, q : D \to S(\mathbb{C})\rangle$$

*of $T_{\omega_1,\omega}(\mathbf{p})$, of cardinality $2^{\aleph_0}$, such that there exists*

$$\{\tau_i\}_{i=1}^m \subset D,$$

*such that, for all $i = 1, ..., m$,*

$$q(g_1 x_i, ..., g_n x_i) \in Z_g$$

*corresponds to $[x_{i,g}]_g \in \Gamma_g\backslash X^{+}$ for all tuples $g = (g_1, ..., g_n)$.*

**Proposition 4.2** *Let $x_1, ..., x_m \in X^{+}$ be Hodge-generic points in distinct $G^{\mathrm{ad}}(\mathbb{Q})^{+}$-orbits. Let*

$$\mathbf{q} := \langle\langle D, G(\mathbb{Q})\rangle, \langle S(\mathbb{C}), \{Z_i\}_{i\in\mathbb{N}}\rangle, q : D \to S(\mathbb{C})\rangle$$

*be a model of $T_{\omega_1,\omega}(\mathbf{p})$, of cardinality $2^{\aleph_0}$, and let $x'_1, ..., x'_m \in D$ be such that $q(x'_i) = p(x_i)$. Consider the type of the tuple $(x'_1, ..., x'_m)$ in $\mathbf{q}$ over the field $K$ obtained from $E^{\mathrm{ab}}$ by adjoining the co-ordinates of the $p(x_1), ..., p(x_m)$. The set of types obtained in this way, varying over all such models of $T_{\omega_1,\omega}(\mathbf{p})$, is either finite or of cardinality $2^{\aleph_0}$.*

**Proof.** By Proposition 2.8 and the fact that the $x_i'$ are in distinct $G^{\mathrm{ad}}(\mathbb{Q})^+$-orbits, the type described in the proposition is determined by the union of the algebraic types of the points

$$(q(g_1x_1'),...,q(g_nx_1'),q(g_1x_2'),...,q(g_nx_2'),...,q(g_1x_m'),...,q(g_nx_m')) \in Z_g^m,$$

over all tuples $g = (g_1,...,g_n)$ (as defined previously). Each of these is determined by the minimal algebraic subvariety of $Z_g^m$, defined over $K$, containing the given point; it corresponds to a subset of the fibre over the base point in $S^m(\mathbb{C})$. Therefore, the number of possible subvarieties is bounded by the index of $\mathrm{Aut}(\mathbb{C}/K)$ in $(\Gamma_{\alpha g}\backslash\Gamma)^m$. However, by Proposition 4.1, every possible subvariety arises in some model of $T_{\omega_1,\omega}(\mathbf{p})$. For each successive tuple containing one before it, this index increases by an integer multiple. Thus, either the number stabilises, or continues to increase in, at least, multiples of two. $\square$

**Theorem 4.3 (Keisler)** *If an $\mathcal{L}_{\omega_1,\omega}$-sentence $\Sigma$ is $\aleph_1$-categorical then the set of complete $n$-types realisable in models of $\Sigma$ is at most countable.*

**Corollary 4.4** *Let $x_1,...,x_m \in X^+$ be Hodge-generic in distinct $G^{\mathrm{ad}}(\mathbb{Q})^+$-orbits and let $K$ be the field obtained from $E^{\mathrm{ab}}$ by adjoining the coordinates of*

$$z := (p(x_1),...,p(x_m)) \in S^m(\mathbb{C}).$$

*If $T_{\omega_1,\omega}(\mathbf{p})$ is $\aleph_1$-categorical then the corresponding homomorphism*

$$\mathrm{Aut}(\mathbb{C}/K) \to \overline{\Gamma}^m$$

*has finite index.*

# 5 Categoricity

Consider a bi-typal variety $S$, as defined in (1.1), and the corresponding $\mathcal{L}$-structure $\mathbf{p}$, as described in (1.2). We define a pregeometry (see [2], §1) on the class of $\mathcal{L}$-structures that are models of $T_{\omega_1,\omega}(\mathbf{p})$ by defining, for each such model

$$\mathbf{q} := \langle\langle D, G^{\mathrm{ad}}(\mathbb{Q})^+\rangle, \langle S(F), \{V_i\}_{i\in\mathbb{N}}\rangle, q : D \to S(F)\rangle,$$

a closure operator $\mathrm{cl_q} := \mathrm{q}^{-1} \circ \mathrm{acl} \circ \mathrm{q}$, where acl denotes taking all points in $S(F)$ with coordinates in the algebraic closure of the coordinates of the points in question.

We note that the pregeometry axioms I.1, I.2, I.3 and the $\aleph_0$-homogeneity axiom II.1 of [2], §1 are easily verified. Thus, we are left to check axiom II.2:

**Lemma 5.1** *Suppose that*

$$\mathbf{q} := \langle\langle D, G^{\mathrm{ad}}(\mathbb{Q})^+\rangle, \langle S(F), \{V_i\}_{i\in\mathbb{N}}\rangle, q : D \to S(F)\rangle$$

*and*

$$\mathbf{q}' := \langle\langle D, G^{\mathrm{ad}}(\mathbb{Q})^+\rangle, \langle S(F), \{V_i\}_{i\in\mathbb{N}}\rangle, q' : D \to S(F)\rangle$$

*are models of* $T_{\omega_1,\omega}(\mathbf{p})$ *and* $K \subset \mathbb{C}$ *is an extension of a countable algebraically closed field or the field obtained from* $E^{\mathrm{ab}}$ *by adjoining the coordinates of all special points in* $S(F)$*. Assume that* $K$ *is generated by the coordinates of a tuple of Hodge-generic points*

$$(x_1, ..., x_{m-1}) \in D^{m-1} \subset \mathbf{q}$$

*in distinct* $G^{\mathrm{ad}}(\mathbb{Q})^+$*-orbits and suppose that* $(x'_1, ..., x'_{m-1})$ *is a tuple of points in* $D \subset \mathbf{q}'$ *with the same quantifier-free type over* $K$*. Suppose that, for any Hodge generic point* $\tau \in D$*, disjoint from the* $G^{\mathrm{ad}}(\mathbb{Q})^+$*-orbits of the* $x_i$*, and* $L$ *the extension of* $K$ *generated by its coordinates in* $S(F)$*, the homomorphism*

$$\mathrm{Aut}(F/L) \to \overline{\Gamma}^m,$$

*corresponding to the tuple*

$$(q(x_1), ..., q(x_{m-1}), q(\tau)) \in S^m(F)),$$

*has finite index. Then, there exists a* $\tau' \in D \subset \mathbf{q}'$ *such that the extended tuples have the same quantifier-free types over* $K'$*.*

**Proof.** The property of the above homomorphism being of finite index implies that there exists a tuple $g = (g_1, ..., g_n)$ such that the homomorphism

$$\mathrm{Aut}(F/L) \to \overline{\Gamma_g}^m$$

is surjective. Hence, a valid $\tau'$ will be any realisation of the quantifier-free type of $(q(g_1\tau), ..., q(g_n\tau))$ in $S^m(F)$. However, by Corollary 2.7, such a realisation always exists. □

**Corollary 5.2** *Under the assumptions in Lemma 5.1, $T_{\omega_1,\omega}(\mathbf{p})$ has a unique model in cardinality $\aleph_1$, up to isomorphism.*

**Proof.** See [2], Corollary 2.2. $\square$

We are now able to achieve our objective stated in (1.3):

**Theorem 5.3** *Under the assumptions in Lemma 5.1, $T^{\infty}_{\omega_1,\omega}(\mathbf{p})$ is $\kappa$-categorical for all uncountable cardinalities $\kappa$.*

**Proof.** By Lemma 5.1, the model $\mathbf{p}$ with the pregeometry $\mathrm{cl}_{\mathrm{p}}$ is a quasiminimal pregeometry structure (as in [1], §2) and $\mathcal{K}(\mathbf{p})$ is a quasiminimal class. So, by the main result (Theorem 2.2) of [1], $\mathcal{K}(\mathbf{p})$ has a unique model (up to isomorphism) in each infinite cardinality and, in particular, $\mathcal{K}(\mathbf{p})$ contains a unique structure $\mathbf{p_0}$ of cardinality $\aleph_0$. Let $\mathcal{K}$ be the class of models of $T^{\infty}_{\omega_1,\omega}(\mathbf{p})$; it is clear that $\mathcal{K}(\mathbf{p}) \subset \mathcal{K}$ since $\mathbf{p} \in \mathcal{K}$. Thus, to prove the theorem we need to show that $\mathcal{K} \subset \mathcal{K}(\mathbf{p})$.

By Lemma 5.1, $\mathcal{K}$ has a unique model $\mathbf{q_0}$ of cardinality $\aleph_0$ and, as a class of models of an $\mathcal{L}_{\omega_1,\omega}$-sentence, it is an abstract elementary class with Lowenheim-Skolem number $\aleph_0$. So, by the downward Lowenheim-Skolem theorem (in $\mathcal{K}$), every model in $\mathcal{K}$ is a direct limit of copies of the unique model of cardinality $\aleph_0$ (with elementary embeddings as morphisms). Finally, all the embeddings in $\mathcal{K}$ are closed with respect to the pregeometry. Therefore, $\mathcal{K} = \mathcal{K}(\mathbf{q_0}) = \mathcal{K}(\mathbf{p_0}) = \mathcal{K}(\mathbf{p})$. $\square$

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